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# String Effects in the Wilson Loop: a high precision numerical test

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**Abstract**

We test numerically the effective string description of the infrared limit of lattice gauge theories in the confining regime. We consider the $3d$ $\mathbb{Z}_2$ lattice gauge theory, and we define ratios of Wilson loops such that the predictions of the effective string theory do not contain any adjustable parameters. In this way we are able to obtain a degree of accuracy high enough to show unambiguously that the flux–tube fluctuations are described, in the infrared limit, by an effective bosonic string theory.

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# 1 Introduction

It is widely believed that the confining regime of Yang-Mills theory is described by some kind of effective string model [1]. It is the aim of this paper to present some numerical evidence supporting this conjecture and, what is most important, to study the nature of the effective string which is actually realized in the infrared regime of Lattice Gauge Theories (LGTs).

This conjecture has by now a very long history. It originates from two independent observations. The first one is of phenomenological nature, and dates before the formulation of QCD. It is related to the observation that the linearly rising Regge trajectories in meson spectroscopy can be easily explained assuming a string-type interaction between the quark and the antiquark. This observation was at the origin of a large amount of papers which tried to give a consistent quantum description of strings.

In the same years, a completely independent observation shed some new light on the problem of the confinement of quarks. It was shown that in the strong coupling limit of pure LGTs the interquark potential rises linearly, and that the chromoelectric flux lines are confined in a thin "string like" flux tube [2]; however this approximation is plagued by lattice artifacts which make it inadequate for the continuum theory. Some clear indications were later found that the vacuum expectation value of Wilson loops could be rewritten as a string functional integral even in the continuum [3, 4, 5]. This led to conjecture that there exists an exact duality between gauge fields and strings [3]. Although this approach has been studied extensively (see e. g. [6, 7, 8, 9]), such an exact duality has been proven only in $2d$ QCD in the large $N$ limit. In this paper we do not assume this strong conjecture, but a very mild version of it, which states that the behavior of large Wilson loops is described in the infrared limit by an effective two-dimensional field theory which accounts for the string-like properties of long chromoelectric flux tubes. We shall refer to such a $2d$ field theory in the following as the "effective string theory".

Let us briefly outline the pattern of this paper. In the next section we shall discuss some general features of the effective string theory. In sect. 3 we shall evaluate in this framework the finite size correction for the expectation values of Wilson loops. This theoretical prediction will then be compared with the result of a set of high precision Monte Carlo simulations in sect.5, while in sect.4 we shall describe the $3d$ $\mathbb{Z}_2$ gauge model and the algorithms that we used to measure these expectation values: a gauge version of the microcanonical demon algorithm and a non-local cluster algorithm applied to the dual spin model. Finally sect. 6 will be devoted to some concluding remarks.

# 2 Some features of the effective string theory

The aim of this section is to give some introductory material so as to make this paper self–contained. We shall first show how the effective string emerges in LGT (sect. 2.1) and discuss the peculiar finite size effects that it induces in the expectation value of large Wilson loops (sect. 2.2). We shall then address the question of the "string universality" namely the fact that these effects show a substantial independence on the gauge group (sect. 2.3). This will allow us to concentrate on the simplest possible non trivial gauge theory, namely the $3d$ gauge Ising model to test the effective string theory. By means of a duality transformation we shall then use some results from the physics of interfaces to obtain some preliminary expectation on the nature of the effective string model (sect. 2.4). These expectations will then be confirmed in the rest of the paper by a direct comparison with the results of Monte Carlo simulations.

## 2.1 Flux Tubes in the Rough Phase

The confining regime of LGTs contains in general two phases: the strong coupling phase and the rough phase. The two are separated by the so called "roughening transition" which is the point in which the strong coupling expansion of the Wilson loop ceases to converge [10, 11]. These two phases are related to two different behaviors of the quantum fluctuations of the flux tube around its equilibrium position [11]. In the strong coupling phase, these fluctuations are massive, while in the rough phase they become massless and hence survive in the continuum limit. This fact has several consequences:

(a) The flux–tube fluctuations can be described by a suitable two-dimensional massless quantum field theory (QFT), where the fields describe the transverse displacements of the flux tube. This quantum field theory is expected to be very complicated and will contain in general non renormalizable interaction terms [11, 12]. However, exactly because these interactions are non-renormalizable, their contribution becomes negligible in the infrared limit (namely for large Wilson loops). In this infrared limit the QFT becomes a conformal invariant field theory (CFT).

(b) The massless quantum fluctuations delocalize the flux tube which acquires a nonzero width, which diverges logarithmically as the interquark distance increases [13, 14].

(c) The quantum fluctuations give a non-zero contribution to the interquark potential, which is related to the partition function of the above $2d$ QFT. Hence if the $2d$ QFT is simple enough to be exactly solvable (and this is in general the case for the CFT in the infrared limit) also these contributions can be evaluated exactly.

(d) In the simplest case, this CFT is simply the two dimensional conformal field theory of $(d-2)$ free bosons ($d$ being the number of spacetime dimensions of the original gauge model); its exact solution will be discussed below.

### 2.2 Finite Size Effects

The feature of the effective string description which is best suited to be studied by numerical methods is the presence of finite–size effects. Wilson loops are classically expected to obey the famous area-perimeter-constant law. This means that the expectation value of a Wilson loop of size $R \times T$ must behave, as $R$ and $T$ vary, as:

$$< W(R,T) >= e^{-\sigma RT + p(R+T) + k} \ . \tag{1}$$

This law is indeed very well verified in the strong coupling regime (before the roughening transition). However in the rough phase it must be modified. One must multiply it by the partition function of the $2d$ QFT describing the quantum fluctuations of the flux tube. As we have seen before, this QFT in the infrared limit becomes a two dimensional CFT, and its partition function $Z_q(R,T)$ can be evaluated exactly. (See *e.g.* Ref. [15] for a comprehensive review on CFTs). In the next section we shall describe in detail an example of this type of calculations. Eq. (1) in the rough phase becomes:

$$< W(R,T) >= e^{-\sigma RT + p(R+T) + k} Z_q(R,T) \ . \tag{2}$$

In the limit $T >> R$ (which is the relevant one to extract the interquark potential) the partition function of the most general CFT yields:

$$\lim_{T\to\infty} \frac{1}{T} \log Z_q(R,T) = \frac{c\pi}{24R} \quad , \tag{3}$$

where $c$ is the central charge of the CFT. In the simplest possible case, namely when the CFT describes a collection of $n$ free bosonic fields, we have $c = n$. Thus for the free boson realization of the effective string theory, we find $c = d-2$. This is the result obtained by Lüscher, Symanzik and Weisz in [11].

The interquark potential is thus given by:

$$V(R) = -\lim_{T\to\infty} \frac{1}{T} \log\langle W(R,T)\rangle = \sigma R - \frac{c\pi}{24R} \quad . \tag{4}$$

The $1/R$ term in the potential is the finite size effect mentioned above; it is completely due to the quantum fluctuations of the flux tube and, if unambiguously detected, it represents a strong evidence (the strongest we have) in favor of the effective string picture discussed above. Moreover if the measurement is precise enough we can in principle extract numerically the value of $c$ and thus

select which kind of effective string model describes the infrared regime of the LGT under examination.

In LGTs with continuous gauge groups the interquark potential has a further term which is due to the one gluon exchange. It can be evaluated perturbatively, and it exists only in the ultraviolet regime, namely for small Wilson loops. It has the form of a Coulombic interaction, hence in (3+1) dimensions it is also of the type $1/R$ and can shadow the contribution coming from the flux tube fluctuations. Even if it holds only in the perturbative regime, we cannot fix a sharp threshold after which it disappears, so it could well be that, in the set (of large Wilson loops) from which we extract our data we find a superposition of the two terms. There are two ways to avoid this problem:

(a) Study LGT in three dimensions where the perturbative term has a $\log R$ form instead of $1/R$, and does not mix up with the string contribution.

(b) Study Wilson loops with comparable values of $T$ and $R$. In this case, the whole functional form of the two interaction terms becomes important. These are completely different and thus can be separated.

Actually the latter option is anyway much more interesting than the study of the $T \to \infty$ limit only, since the whole functional form of the CFT partition function is richer, and allows more stringent numerical tests. In this paper we shall follow this strategy and study always Wilson loops with comparable values of $T$ and $R$. This choice will magnify the finite size effects and will simplify our numerical work.

Since the beginning of eighties several numerical works have been done to study this problem. The main results can be summarized as follows:

(a) A $1/R$ term exists in the potential. In the case of (3+1) LGT with continuous gauge group it can be separated form the perturbative term.

(b) The central charge has not been measured with good precision, however the numbers seem to be in reasonable agreement with the $c = d - 2$ prediction of Lüscher Symanzik and Weisz. The lack of precision, and the consequent impossibility to determine which kind of effective string is actually realized in the infrared regime of LGT was one of the main motivations of the present work.

(c) The same correction is found in very different LGTs, ranging from the $3d$ Ising gauge model to the $4d$ SU(3) model. This remarkable universality is an important feature of these finite size effects of the effective string description.

### 2.3 String Universality

As a matter of fact not only the string corrections, but also other features of the infrared regime of LGTs in the confining phase display a high degree of

universality, namely they seem not to depend on the choice of the gauge group. This is the case for instance of the ratio between the critical temperature and the square root of the string tension, or the behavior of the spatial string tension above the deconfinement transition. Recently this same universality has been evidentiated in the pattern of the glueball states for various three dimensional gauge models [16]. All these examples show a substantial independence on the gauge group and a small and smooth dependence on the number of spacetime dimensions.

This "experimental fact" has a natural explanation in the context of an effective string model: even if in principle different gauge models could be described by different string theories[1], in the infrared regime, as the interquark distance increases all these different string theories flow toward the common fixed point which is not anomalous and corresponds, in the simplest case discussed above, to the two dimensional conformal field theory of $(d-2)$ free bosons. Also the small dependence on the number of spacetime dimensions of the theory is well predicted by the effective string theory.

A very important consequence of this universality is that it allows us to study the universal infrared behavior of Wilson loops in the case of the three dimensional Ising or $\mathbb{Z}_2$ gauge model, which is the simplest non-trivial LGT and allows high precision Monte Carlo simulations with a relatively small amount of CPU time. We already used this opportunity in some previous works on the subject, and shall again use it in the present paper. The simplicity of the model and the construction of a new very powerful algorithm to simulate it will allow us to reach a very high degree of precision, never reached before even in the context of the Ising model. This will allow us to test the predictions of the effective string theory with unprecedented confidence and to determine unambiguously the nature of the effective string which describes the infrared regime of LGT.

### 2.4 Interfaces

The physics of the Wilson loop is closely related to the behavior of interfaces in spin models, which is known to be described by an effective string theory, the only difference being in the boundary conditions.

Indeed the most precise numerical test of the effective string theory up to now has been performed in the physics of interfaces of the three dimensional Ising model [12]. The gauge Ising model is transformed by duality in the spin Ising model and the Wilson loops are related to interfaces in the context of the spin model[2]. The string tension is dual to the interface tension. The effective

[1] Notice however that recently Polyakov [8] has put forward the conjecture that, even at the fundamental level all the Yang Mills models are described by one and the same string theory.

[2] The term "roughening transition" was actually first introduced in the context of the physics of interfaces and only later it was also used in the study of Wilson loops.

string model is known, in the context of interface physics as the capillary wave model. According to this model interfaces in the rough phase are described by a $2d$ quantum field theory of a free, massless bosonic field plus an infinite set of non-renormalizable interaction terms; these can be written explicitly in terms of the bosonic field and of its derivatives. In the infrared limit the capillary wave model becomes a bosonic free field theory hence a CFT with $c = 1$. In a set of recent papers this prediction has been compared with some Monte Carlo result on the interface free energy in the $3d$ Ising model and a remarkable agreement has been found.
Moreover, this same effective description of interfaces has been derived analitically in the $3d$ $\lambda\phi^4$ quantum field theory [17]. The high degree of universality of this picture is shown by the fact that also interfaces in the $3d$ three-state Potts model are accurately described by the same effective theory [18].
It is therefore quite natural to conjecture that Wilson loops in the infrared limit are described by the same CFT. The aim of this work is precisely to verify whether the effective string description that has been shown to describe interface physics in $3d$ spin models can be extended to the Wilson loop behavior in the corresponding gauge model, once the necessary modifications in the boundary conditions have been taken into account. It turns out that even in this case the quantum fluctuations in the infrared limit are described by a $c = 1$ CFT.

# 3 The prediction of the effective string theory

In this section we review the computation of the effective string theory prediction for the behavior of the Wilson loop in the infrared limit: we will show how a rather large class of bosonic string theories reduce in this limit to the theory of $d-2$ free massless scalar fields, and we will compute the partition function $Z_q(R,T)$ of this CFT. This expression will be plugged in Eq. (2) to obtain the prediction of the effective string theory for the behavior of large Wilson loops [19].

Let us consider for example the Nambu string action, given by the area of the world–sheet:

$$S = \sigma \int_0^T d\tau \int_0^R d\varsigma \sqrt{g} \quad , \tag{5}$$

where $g$ is the determinant of the two–dimensional metric induced on the world–sheet by the embedding in $R^d$:

$$\begin{aligned} g = \det(g_{\alpha\beta}) \quad &= \quad \det\ \partial_\alpha X^\mu \partial_\beta X^\mu \quad . \\ & \quad\ (\alpha, \beta = \tau, \varsigma,\ \mu = 1, \ldots, d) \end{aligned} \tag{6}$$

and $\sigma$ is the string tension.

The reparametrization and Weyl invariances of the action (5) require a gauge

choice for quantization. We choose the "physical gauge"

$$\begin{aligned} X^1 &= \tau \\ X^2 &= \varsigma \end{aligned} \tag{7}$$

so that $g$ is expressed as a function of the transverse degrees of freedom only:

$$\begin{aligned} g &= 1 + \partial_\tau X^i \partial_\tau X^i + \partial_\varsigma X^i \partial_\varsigma X^i \\ &\quad + \partial_\tau X^i \partial_\tau X^i \partial_\varsigma X^j \partial_\varsigma X^j - (\partial_\tau X^i \partial_\varsigma X^i)^2 \\ &\quad (i = 3, \ldots, d) \ . \end{aligned} \tag{8}$$

The fields $X^i(\tau, \varsigma)$ satisfy Dirichlet boundary conditions on $M$:

$$X^i(0,\varsigma) = X^i(T,\varsigma) = X^i(\tau,0) = X^i(\tau,R) = 0 \ \ . \tag{9}$$

Due to the Weyl anomaly this gauge choice can be performed at the quantum level only in the critical dimension $d = 26$. However, the effect of the anomaly is known to disappear at large distances [20], which is the region we are interested in.

Expanding the square root in Eq. (5) we obtain, discarding terms of order $X^4$ and higher

$$S = \sigma RT + \frac{\sigma}{2} \int d^2\xi X^i(-\partial^2) X^i \tag{10}$$

$$\partial^2 = \partial_\tau^2 + \partial_\varsigma^2 \ . \tag{11}$$

It is easy to see that this expansion of the action corresponds, for the partition function, to an expansion in powers of $(\sigma RT)^{-1}$. Therefore the action (10) describes the infrared limit of the model defined by Eq. (5), and will be relevant to the physics of large Wilson loops. While the choice of the action (5) is not universal, a large class of bosonic effective string models reduce to this CFT in the infrared limit.The contribution of the fluctuations of the flux–tube to the Wilson loop expectation value in the infrared limit will be the partition function of our CFT, given by

$$Z_q(R,T) \propto \left[\det(-\partial^2)\right]^{-\frac{d-2}{2}} \ \ . \tag{12}$$

The determinant must be evaluated with Dirichlet boundary conditions.

The spectrum of $-\partial^2$ with Dirichlet boundary conditions is given by the eigenvalues

$$\lambda_{mn} = \pi^2 \left( \frac{m^2}{T^2} + \frac{n^2}{R^2} \right) \tag{13}$$

corresponding to the normalized eigenfunctions

$$\psi_{mn}(\xi) = \frac{2}{\sqrt{RT}} \sin \frac{m\pi\tau}{T} \sin \frac{n\pi\varsigma}{R} \ . \tag{14}$$

The determinant appearing in Eq. (12) can be regularized with the $\zeta$-function technique: defining

$$\zeta_{-\partial^2}(s) \equiv \sum_{mn=1}^{\infty} \lambda_{mn}^{-s} \tag{15}$$

the regularized determinant is defined through the analytic continuation of $\zeta'_{-\partial^2}(s)$ to $s=0$:

$$\det(-\partial^2) = \exp\left[-\zeta'_{-\partial^2}(0)\right] \quad . \tag{16}$$

The series in Eq. (15) can be transformed, using the Poisson summation formula, to read

$$\zeta_{-\partial^2}(s) = -\frac{1}{2}\left(\frac{R^2}{\pi^2}\right)^s \zeta_R(2s) + \frac{\sqrt{\pi} Im\tau \Gamma(s-1/2)}{2\Gamma(s)}\left(\frac{R^2}{\pi^2}\right)^s \zeta_R(2s-1)$$
$$+\frac{2\sqrt{\pi}}{\Gamma(s)}\left(\frac{T^2}{\pi^2}\right)^s \sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\left(\frac{\pi p}{n Im\tau}\right)^{s-1/2} K_{s-1/2}(2\pi p n Im\tau) \tag{17}$$

where $\tau = iT/R$, $\zeta_R(s)$ is the Riemann $\zeta$ function and $K_\nu(x)$ is a modified Bessel function. The derivative $\zeta'_{-\partial^2}(s)$ can be analytically continued to $s=0$ where it is given by

$$\zeta'_{-\partial^2}(0) = \log(\sqrt{2R}) - \frac{i\pi\tau}{12} - \sum_{n=1}^{\infty}\log(1-q^n) \tag{18}$$

where we have defined

$$q \equiv e^{2\pi i\tau} \quad . \tag{19}$$

Introducing the Dedekind $\eta$-function

$$\eta(\tau) = q^{1/24}\Pi_{n=1}^{\infty}(1-q^n) \tag{20}$$

we obtain finally

$$\det(-\partial^2) = \exp[-\zeta'_{-\partial^2}(0)] = \frac{\eta(\tau)}{\sqrt{2R}} \tag{21}$$

and

$$Z_q(R,T) \propto \left[\frac{\eta(\tau)}{\sqrt{R}}\right]^{-\frac{d-2}{2}} \quad . \tag{22}$$

We have checked numerically that a lattice regularization of the determinant gives the same result.

Substituting in Eq. (2) we obtain [19]

$$< W(R,T) >= e^{-\sigma RT + p(R+T)+k}\left[\frac{\eta(\tau)}{\sqrt{R}}\right]^{-\frac{d-2}{2}} \quad . \tag{23}$$

This is the prediction that we are going to compare with numerical results. It is important to appreciate the predictive power of the model: the inclusion of the fluctuation contribution does not add any new adjustable parameters.

# 4 The $\mathbb{Z}_2$ gauge model and the simulation algorithms

## 4.1 The Model

The $3d$ $\mathbb{Z}_2$ gauge model in a cubic lattice is defined by the partition function

$$Z_{gauge}(\beta) = \sum_{\{\sigma_l=\pm 1\}} \exp\left(-\beta S_{gauge}\right) \ . \tag{24}$$

The action $S_{gauge}$ is a sum over all the plaquettes of a cubic lattice,

$$S_{gauge} = -\sum_{\Box} \sigma_{\Box} \quad , \quad \sigma_{\Box} = \sigma_{l_1}\sigma_{l_2}\sigma_{l_3}\sigma_{l_4}, \tag{25}$$

where $\sigma_l \in \{1, -1\}$ are Ising variables located in the links of the lattice.

This model can be translated into the $3d$ spin Ising model by the usual Kramers-Wannier duality transformation

$$Z_{gauge}(\beta) \qquad \propto Z_{spin}(\tilde{\beta}) \tag{26}$$

$$\tilde{\beta} \quad = -\tfrac{1}{2}\log\left[\tanh(\beta)\right] \quad , \tag{27}$$

where $Z_{spin}$ is the partition function of the Ising model in the dual lattice:

$$Z_{spin}(\tilde{\beta}) = \sum_{s_i=\pm 1} \exp(-\tilde{\beta} H_1(s)) \tag{28}$$

with

$$H_1(s) = -\sum_{\langle ij\rangle} J_{\langle ij\rangle} s_i s_j \tag{29}$$

where $i$ and $j$ denote nodes of the dual lattice and the sum is extended to the links $\langle ij\rangle$ connecting the nearest-neighbor sites. In general the couplings $J_{\langle ij\rangle}$ are fixed to the value $+1$ for all the links, but we shall use the freedom to flip some of them in the following. The model is known to have a roughening transition located at $\beta_r = 0.47542(1)$[27], and a deconfinement transition at $\beta_d = 0.7614133(22)$[28]. We performed our Monte Carlo simulations at three different values of the coupling constant $\beta$, all located in the rough phase and close enough to the deconfinement point to be well inside the scaling region. To avoid systematic errors due to the choice of the simulation algorithm we used two completely different alghoritms. The first one is an improved version of the microcanonical demon algorithm, it works directly in the $\mathbb{Z}_2$ gauge model and we shall describe it in sect. 4.2 below. The second one is a non-local cluster algorithm and was used to simulate the dual spin Ising model (sect. 4.3). The perfect agreement between the results obtained with these two different methods is a strong consistency check of the reliability of our data.

## 4.2 The Demon Algorithm

In our simulation the microcanonical demon algorithm of Ref. [21] was combined with a particularly efficient canonical update of the demons [22] in order to obtain the canonical ensemble of the gauge model. This algorithm was implemented in the multispin coding technique. This means that each of the 64 bits of a word is used to store a link variable. The update is implemented with bit-operations. Hence 64 systems can be updated in parallel. Note that also the measurement of the Wilson loops is implemented in the multispin coding technique.

The multispin coding implementation combined with the use of the microcanonical demon algorithm, which saves pseudo random numbers provides us with a performance increase compared with a naive implementation of the Metropolis algorithm of about a factor of 100.

The update of a single link variable took about $2.6 \times 10^{-8}$ *sec* on a DEC250 4/266 workstation, which is rated at 5.18 SPECint95, 6.27 SPECfp95 and 53.96 LINPACK 100 X 100 MFLOPS.

For moderate correlation lengths, as discussed in this paper, the microcanonical demon algorithm should provide superior performances compared to a cluster-update [23].

For a detailed discussion see Ref. [16] where the same algorithm was used.

## 4.3 Non-local Cluster Algorithm

Using the duality transformation it is possible to build up a one-to-one mapping of physical observables of the gauge system into the corresponding spin quantities. For instance, the vacuum expectation value of a Wilson loop $W(C)$ can be expressed in terms of spin variables as follows. First, choose a surface $\Sigma$ bounded by $C$: $\partial\Sigma = C$; then take $J_{\langle ij \rangle} = -1$ for all those links intersecting $\Sigma$ and denote with $H_{-1}(s)$ the Ising Hamiltonian with this choice of couplings. The new Ising partition function

$$Zw_{spin}(\tilde{\beta}) = \sum_{s_i=\pm 1} \exp\left(-\tilde{\beta} H_{-1}(s)\right) \tag{30}$$

describes a vacuum modified by the Wilson loop $W(C)$; we call it the W-vacuum (WV). It is easy to see that

$$\langle W(\partial\Sigma)\rangle_{gauge} = \frac{Zw_{spin}}{Z_{spin}} = \langle \prod_{\langle ij \rangle \in \Sigma} \exp(-2\tilde{\beta} s_i s_j)\rangle_{spin} \quad , \tag{31}$$

where the product is over all the dual links intersecting $\Sigma$.

For large Wilson loops there is some problem in calculating the above expectation value, because each term inside the brackets is the product of factors which can be very large or very small, then one expects large fluctuations. There

is however a new method where this difficulty is overcome by adapting a new procedure first applied by one of us [24] to the calculation of the surface tension. The idea is to consider the sign $w = \pm 1$ of the couplings settling the presence or the absence of the Wilson loop in the vacuum as a dynamical variable and sum over it. Thus the new partition function is now

$$Z = \sum_{w=\pm 1} \sum_{s_i=\pm 1} \exp\left(-\tilde{\beta} H_w(s)\right) \quad . \tag{32}$$

The fraction of configurations with the Wilson loop is given by the ratio

$$\frac{Zw}{Z} = \frac{\sum_{w=\pm 1} \sum_{s_i=\pm 1} \exp(-\tilde{\beta} H_w(s))\delta_{w,-1}}{Z} = \langle \delta_{w,-1} \rangle \tag{33}$$

while the fraction of configurations corresponding to the ordinary vacuum is $\langle \delta_{w,1} \rangle$. The comparison with Eq. (31) yields

$$\langle W(\partial\Sigma) \rangle_{gauge} = \frac{\langle \delta_{w,-1} \rangle}{\langle \delta_{w,1} \rangle} \quad , \tag{34}$$

hence the evaluation of this vacuum expectation value is brought back to mediate over quantities which take only 0 or 1 values.

A great advantage of mapping a gauge observable into one of a spin system is that it can be used, in place of the usual Metropolis or heat-bath methods, a non local cluster updating algorithm [25], which has been proven very successful in fighting critical slowing down. For instance, the Swendsen-Wang cluster algorithm [25] is made of two steps:

- Some bonds are deleted with the following rule: the bond $\langle ij \rangle$ is deleted with probability
$$p = \begin{cases} \exp(-2\tilde{\beta}), & \text{if } s_i = s_j \\ 1 \quad , & \text{if } s_i \neq s_j \end{cases} \tag{35}$$
In this way the lattice is split into clusters of sites connected by the remaining (or frozen) bonds; these form the so called Kasteleyn- Fortuin (KF) clusters [26].
- These clusters are randomly flipped with probability one half.

This updating algorithm provides us with a very efficient method to estimate $\langle W(\partial\Sigma) \rangle_{gauge}$. It is sufficient to look at the KF clusters which intersect $\Sigma$ in each MC configuration: if at least one of these clusters is topologically linked to the loop $C = \partial\Sigma$ with an odd winding number, then the MC configuration is incompatible with the presence of the Wilson loop (hence $\delta_{w,-1} = 0$).

On the contrary, if all these clusters are not linked to $C$, the configuration is compatible with the presence of $W(C)$: the clusters intersected by $\Sigma$ are split

into an upper and a lower part. We can flip all the spins of the lower parts and the signs of the couplings $J_{\langle ij \rangle}$ cut by $\Sigma$, transforming a configuration of the ordinary vacuum into one of the W-vacuum. Conversely, if the MC configuration is one of the W-vacuum, there is no cluster linked to $C$ and the configuration can be always transformed, within the same flipping procedure, into one of the ordinary vacuum. Because of such a property of the WV configurations, it is not actually necessary to flip the $J_{\langle ij \rangle}$ couplings, and the evaluation of $\langle W(C) \rangle$ is reduced to study the winding number of the KF clusters around the loop $C$ [30] : denoting with $N_W$ the number of MC configurations compatible with a W-vacuum and with $N$ the total number of iterations we have simply

$$\langle W(\partial\Sigma) \rangle = \frac{N_W}{N} \quad . \tag{36}$$

# 5 Comparison with Monte Carlo data

In this section we compare the prediction of the effective string theory with Monte Carlo data for the $3d$ $\mathbb{Z}_2$ gauge model. Including the effective string contributions computed in Sec. 2 the functional dependence of the Wilson loop on the sides $R, T$ is, for $d = 3$,

$$\langle W(R,T) \rangle = \exp\left[-\sigma TR + p(T+R) + k\right] \left[\frac{\eta(\tau)}{\sqrt{R}}\right]^{-1/2} \quad . \tag{37}$$

The values of the parameters $\sigma$, $p$ and $k$ are not predicted by the effective string theory. However, we can eliminate $p$ and $k$ by considering ratios of Wilson loops with equal perimeter. Moreover, the value of $\sigma$ can be taken from high precision numerical simulations of the $3d$ Ising spin model, which is dual to the $\mathbb{Z}_2$ gauge model. The string tension of the gauge model corresponds, by duality, to the interface tension of the spin model. Using these values for $\sigma$ we achieve complete independence between the theoretical predictions and the numerical data to be compared.

In particular, defining

$$R(L,n) \equiv \frac{\langle W(L+n, L-n) \rangle}{\langle W(L,L) \rangle} \exp(-n^2\sigma) \tag{38}$$

the string theory prediction for $R(L,n)$ depends only on $t = n/L$:

$$R(L,n) = F(t) = \left[\frac{\eta(i)\sqrt{1-t}}{\eta\left(i\frac{1+t}{1-t}\right)}\right]^{1/2} \tag{39}$$

and does not contain any adjustable parameters.

For $\beta = 0.75245$ we used the non–local cluster algorithm (Subsection 4.3). Non–local algorithms drastically reduce correlations in Monte Carlo time, therefore we could safely perform a measurement for every configuration we produced. On the other hand, the nature of the algorithms is such that only one ratio of Wilson loops can be measured in each run. For each ratio about $5 \cdot 10^6$ iterations were performed.

For $\beta = 0.75202$ and $\beta = 0.75632$ we used the demon algorithm described in Subsection 4.2. The run at $\beta = 0.75632$ on the $80^3$ lattice (from which most of the data reported in this paper were taken) took 6.5 days on a Alpha AXP-3000/400. We performed a measurement after 50 updating sweeps. The integrated autocorrelation time (in units of sweeps) was smaller then 300 for all Wilson loops considered.

For technical reasons in the simulations at $\beta = 0.75202$ and $\beta = 75245$ we chose asymmetric lattices with different sizes $N_s$ and $N_t$ in the space and time directions respectively. For these value of $\beta$ we only measured Wilson loops of sizes $R \times T$ orthogonal to the time direction. $R$ and $T$ were constrained to be: $\xi << R, T << N_s$ where $\xi$ is the bulk correlation length.

A summary of the informations on our data sample is reported in tab.I .

Table 1: *Some informations on the data sample: $\beta$ is the coupling constant of the gauge $\mathbb{Z}_2$ model and $\tilde{\beta}$ the corresponding (dual) coupling for the spin model. "MD" denotes the microcanonical demon algorithm, while "NL" denotes the nonlocal one. $N_t$ and $N_s$ are the lattice sizes. With $N_c$ we denote the total amount of configurations in the run. $\xi$ denotes the bulk correlation length (taken from ref [12] and [16].*

| $\beta$ | $\tilde{\beta}$ | **alg.** | $N_t \times N_s^2$ | $N_c$ | $\xi$ |
|---|---|---|---|---|---|
| 0.75202 | 0.22600 | MD | $70 \times 50^2$ | $2 \cdot 10^6$ | 3.135(9) |
| 0.75245 | 0.22580 | NL | $24 \times 30^2$ | $5 \cdot 10^6$ | 3.170 |
| 0.75632 | 0.22400 | MD | $80^3$ | $1.7 \cdot 10^5$ | 4.64(3) |

For each value of the asymmetry ratio $t$, we have displayed in Tab. 2 the value of $R(L, n)$ corresponding to the largest value of the physical size $\sigma L^2$ available in our data sample. The values of the string tension are taken from Ref. [12] ($\beta = 0.75245$ and $\beta = 0.75632$) and [29] ($\beta = 0.75202$). The comparison with the prediction of the effective string theory gives a reduced $\chi^2$ of 1.2. The same data are plotted in Fig. 1.

Two facts emerge immediately from the data:

a)The contribution of the flux–tube fluctuations are *quantitatively* relevant in the physics of confinement: notice that a simple perimeter-area law would predict all ratios $R(L, n)$ to be equal to one.

Table 2: *Monte Carlo results for the ratios* $R(L,n)$

| $n/L$ | $L$ | $\beta$ | $\sigma$ | $R(L,n)$ | $F(n/L)$ |
|---|---|---|---|---|---|
| 1/5 | 15 | 0.75202 | 0.01023(5) | 1.0104(17) | 1.00881 |
| 1/4 | 20 | 0.75632 | 0.004779(14) | 1.0166(10) | 1.01453 |
| 2/7 | 7 | 0.75245 | 0.009418(61) | 1.0229(24) | 1.01987 |
| 1/3 | 12 | 0.75632 | 0.004779(14) | 1.02881(30) | 1.02901 |
| 3/8 | 8 | 0.75245 | 0.009418(61) | 1.0403(31) | 1.03940 |
| 9/20 | 20 | 0.75632 | 0.004779(14) | 1.0684(23) | 1.06588 |
| 1/2 | 20 | 0.75632 | 0.004779(14) | 1.0911(27) | 1.09153 |
| 3/5 | 25 | 0.75632 | 0.004779(14) | 1.165(11) | 1.17667 |

b) For large enough Wilson loops, the effective bosonic string model, Eq. (12), describes these fluctuations with great accuracy.

It is important to notice that the whole functional form of the string contribution to the Wilson loop must be taken into account to obtain a satisfactory agreement with the numerical data. In particular it is a rather common practice to take into account only the conformal anomaly term, *i.e.* to write

$$Z_q(R,T) \propto q^{-\frac{d-2}{48}} \tag{40}$$

corresponding in our case to

$$R(L,n) = F_0(t) = \exp\left(\frac{\pi}{12}\frac{t}{1-t}\right) \ . \tag{41}$$

It is apparent from Fig.1 that this approximation (dotted line) is not adequate, and can result in serious errors in extracting the string tension from a set of Wilson loop expectation values. The largest part of the discrepancy is due to the $\sqrt{R}$ term in Eq. (22) rather than to the subdominant terms in the expansion of $\eta(\tau)$ in powers of $q$.

If we consider Wilson loops smaller than a threshold size of order $\sigma L^2 \sim 1$, rather significant finite–size effects appear. An example of these is shown in Fig. 2, where we have plotted the values of $R(L,n)$ for a fixed asymmetry ratio $t = 1/2$ and different values of the physical size $\sigma L^2$ of the Wilson loop. The fact that these effects appear to depend on the physical size $\sigma L^2$ rather than on the size expressed in lattice units suggests that they are related to physical effects rather than lattice artifacts. An important lesson can be drawn from these data: the free string model is indeed valid in the infrared limit only. If a data sample consists of too small Wilson loops, one can erroneously conclude that the model is not accurate.

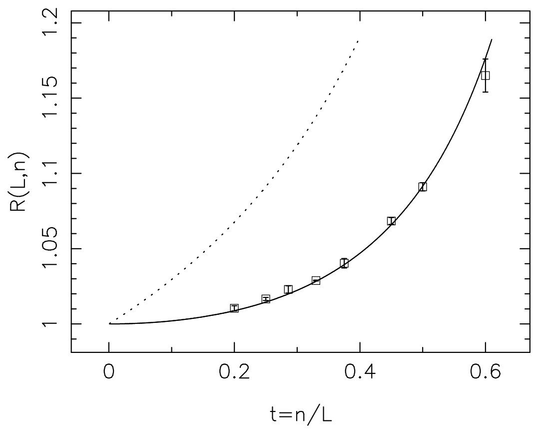


Figure 1: Comparison of the prediction of the free string model with Monte Carlo data for the ratios $R(L, n)$. The solid line is the prediction of the string model, Eq. (39). The dotted line is the same prediction when only the conformal anomaly term is taken into account, Eq. (41).

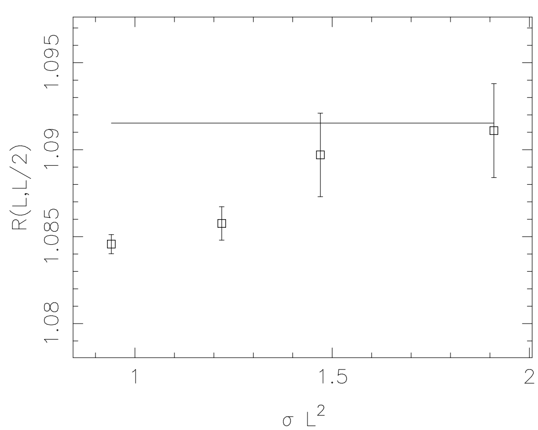


Figure 2: Finite–size effects for small Wilson loop. The prediction of the free string model is $R(L, L/2) = 1.09153\ldots$ (straight line).

## 6 Conclusions

Our present knowledge about the effective string picture of confinement in LGT can be summarized in the following statements:

- The idea of a massless, fluctuating flux–tube joining static sources in the rough phase of LGTs is not only an intuitive physical picture, but has precise and very predictive quantitative consequences, which can be tested with numerical simulations.
- The contribution of the flux–tube fluctuations to the Wilson loop are numerically relevant: failure to take them into account would result in a biased analysis of the data.
- In the infrared regime, for the $3d$ $Z_2$ gauge model, we have shown that the fluctuations are accurately described by an effective bosonic string theory. The high degree of precision attainable in this model allows us this precise identification of the effective string theory.
- However, since it is widely believed that the infrared behavior of LGTs is largely independent of the particular gauge model considered, we conjecture that the same effective string theory will describe the flux–tube fluctuations in a large class of confining LGTs.
- This effective theory is truly an infrared limit: small Wilson loops show rather large finite–size effects. The effective model can be applied only when data for sufficiently large (in physical units) Wilson loops are available.

It is the availability of *precise* data on *large* Wilson loops that makes the precise identification of the effective string theory possible. For example in Ref. [31] a different effective string theory, of fermionic nature, was shown to describe the flux–tube fluctuations as accurately (and in some cases even more accurately) than the bosonic theory. The dramatically increased precision on large Wilson loops allows us now to identify the bosonic model as the most accurate.

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